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A simple competitive graph coloring algorithm II

Charles Dunn[a] and H.A. Kierstead[b]

[a] *Department of Mathematics, Linfield College, 900 SE Baker Street, McMinnville, OR 97128, USA*
[b] *Department of Mathematics and Statistics, Arizona State University, Main Campus P.O. Box 871804, Tempe, AZ 85287-1804, USA*

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Abstract

We consider the following game played on a finite graph G. Let r and d be positive integers. Two players, Alice and Bob, alternately color the vertices of G, using colors from a set of colors X, with $|X| = r$. A color $\alpha \in X$ is a legal color for uncolored vertex v if by coloring v with color α, the subgraph induced by all vertices of color α has maximum degree at most d. Each player is required to color legally on each turn. Alice wins the game if all vertices of the graph are legally colored. Bob wins if there comes a time when there exists an uncolored vertex which cannot be legally colored. We show that if G is a partial k-tree, $r = k + 1$, and $d \geqslant 4k - 1$, then Alice has a winning strategy for this game. In the special case that $k = 1$, this answers a question of Chou, Wang, and Zhu. We also analyze this strategy for other classes of graphs. In particular, we show that there exists a positive integer d such that Alice can win the game on any planar graph if $r = 6$.

1. Introduction

In this paper, we shall analyze a variation of the coloring game originally introduced by Bodlaender in [1]. In the usual version of the coloring game, two players, Alice and Bob, alternate coloring the vertices of a graph G with legal colors from a set of colors X, with $|X| = r$. A color $\alpha \in X$ is legal for an uncolored vertex u, if u has no neighbors already colored with α. Alice wins this game if all of the vertices get colored legally. In this case, the graph has been properly r-colored by the two players. Bob wins if there comes a time in the game when there is an uncolored vertex which cannot be colored legally. The least r such that Alice has a winning strategy is called the *game chromatic number of G*, denoted $\chi_g(G)$. This parameter has proved

E-mail addresses: cdunn@linfield.edu (C. Dunn), kierstead@asu.edu (H.A. Kierstead).

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surprisingly interesting and has been studied in many papers, including [2,8, 11–14,16,19,20]. One of the reasons for this interest is that game chromatic number is well behaved with respect to several natural graph theoretic parameters.

The above formulation facilitates analysis of the problem, but obscures some of the reasons for considering it. While applications have not yet come into focus, it is still useful to speculate. Suppose Alice is trying to color the graph G with the help of an assistant Bob, who is not malevolent, but also is not very smart. Alice will only give the coloring problem half her attention, but still wants a reasonably good result. (Perhaps she is also coloring another graph with the aid of another dumb assistant.) Alternatively, two talented colorers could be coloring the same graph, each with knowledge of a different subset of the edges. Or possibly the two players have different objectives in mind. Perhaps Bob wants to use cheap colors, while Alice still wants to limit the total number of colors.

Here we shall consider a variation, recently introduced by Chou et al. [4], based on the concept of relaxed coloring. A similar variation based on oriented coloring was introduced in [17] and investigated further in [15,16]. A *d-relaxed proper r-coloring* of a graph is an r-coloring such that each color class induces a subgraph with maximum degree at most d. So a 0-relaxed proper r-coloring is just a proper r-coloring. The d-relaxed chromatic number, d-$\chi(G)$, of G is the least integer r such that G has a d-relaxed proper r-coloring. Relaxed colorings, also called defective colorings and improper colorings, have been studied extensively in the literature, including [5–7,10,18]. Chou, Wang and Zhu introduced the *(r,d)-relaxed coloring game*. In this variation, the only change in the rules of the usual coloring game is in the definition of a legal color. As before the players will use a set X of r colors. There is also a fixed natural number d, called the *defect*. A color $\alpha \in X$ is a legal color for an uncolored vertex u if by coloring u with color α, the subgraph induced by all vertices of color α has maximum degree at most d. Again, Alice wins the game if all vertices of the graph are colored legally. In this case the two players have created a d-relaxed proper r-coloring of G. Bob wins if there comes a time when there exists an uncolored vertex which cannot be legally colored. The least r such that Alice has a winning strategy for the (r,d)-relaxed coloring game on G is called the *d-relaxed game chromatic number of* G and is denoted d-$\chi_g(G)$. It is important to note that this notion of a legal color has two facets. If a vertex u is to receive color α, then:

1. u cannot be adjacent to more than d vertices already colored with α.
2. If $v \in N(u)$ has already been colored with α, then v cannot be adjacent to more than $d-1$ vertices already colored α.

It is this second requirement that makes the analysis of the game significantly different than that of the usual coloring game.

The following example is useful for understanding the relaxed coloring game. Let $G=(X,Y,E)$ be the complete bipartite graph $K_{t,t}$. First note that 0-$\chi_g(G) \leqslant 3$: Alice begins by coloring a vertex $x \in X$ with 1. By (1) no vertex in Y can be colored with 1, so 1 will always be a legal color for vertices in X. In the worst case Bob responds by coloring a vertex in X with 2. Then Alice colors a vertex in Y with 3. Now 3 will always be a legal color for any vertex in Y. However, $1\text{-}\chi_g(G) = t$: Without loss of generality, Alice starts the game by coloring a vertex in X with 1. Bob responds by coloring a vertex in Y with 1. By (2) 1 is not a legal color for any vertex. Continuing in this fashion Bob can force Alice to use t colors. Thus increasing the defect does not necessarily make it easier for Alice to win.

In [11], Faigle et al. proved that the $\chi_g(T)$ is at most 4 for any tree T. They provided Alice with an *activation* strategy very roughly described as follows. After Bob colors a vertex, Alice searches for the next vertex to color. Each time her search brings her to an inactive uncolored vertex she activates it and continues her search. One can think of this as leaving a note that the active vertex is dangerous and requires attention the next time that it is encountered. Whenever Alice reaches an active uncolored vertex, she colors it. This technique somehow allows Alice to avoid coloring vertices that would require an immediate response from her, while coloring all dangerous vertices in a timely manner. For a while it seemed that the activation technique was needlessly complicated. Kierstead and Tuza [16] gave an easier proof for trees based on a *separator* strategy, and then extended it to obtain results on chordal graphs. However, in a sequence of papers Guan and Zhu [12], Zhu [19,20], and Kierstead [13] extended the activation technique to outerplanar and planar graphs. In [13] a quite general activation strategy based on careful orderings of the vertices is given. Using this strategy it is possible to prove most known upper bounds on game chromatic number by producing appropriate vertex orderings. In [20] a similar strategy based on careful orientations of the edges of a graph is given. We shall see that both versions of the activation strategy have non-trivial extensions to the relaxed coloring game that can be used to obtain different results.

In their introductory paper [4], Chou et al. showed that there exists a tree T such that $1\text{-}\chi_g(T) > 2$, but that $1\text{-}\chi_g(T) \leqslant 3$ for any tree T. Somewhat ironically they proved the upper bound using a separator strategy, although it is also easy to prove using an activation strategy. They also posed the following question.

Question 1.1 (Chou et al. [4]). Does there exists a defect d such that $d\text{-}\chi_g(T) \leqslant 2$ for every tree T?

In Section 2 we will give a positive answer to this question by using an activation argument to show that $3\text{-}\chi_g(T) \leqslant 2$. We can give two more general solutions to Question 1.1. In Section 3 we will define an extension of the activation strategy given in [13]. In Section 4 we will show that this extension yields the upper bound $(4k-1)\text{-}\chi_g(G) \leqslant k+1$ for any chordal graph G with clique number $\omega(G) \leqslant k+1$. In particular, $3\text{-}\chi_g(T) \leqslant 2$, for any tree T. In Section 5 we will define the notion of k-admissibility and prove that if G is a graph with k-admissibility a then $(a(2k+1)+k)\text{-}\chi_g \leqslant k+1$. For example planar graphs have 5-admissibility at most 8 and outerplanar graphs have 2-admissibility at most 3, so $93\text{-}\chi_g(G) \leqslant 6$ for planar graphs G and $17\text{-}\chi_g(H) \leqslant 3$ for outerplanar graphs H. In [4], Chou et al. showed that $1\text{-}\chi_g(H) \leqslant 6$, for any outerplanar graph H. In Section 6 we will discuss open problems.

In a sequel [9] to this article we will consider a further extension of the activation strategy. We will prove that if G is an (a,b)-pseudo partial k-tree (defined in [20]), then $d\text{-}\chi_g(G) \leqslant k+1$, for sufficiently large d. Using that planar graphs are $(3,8)$-pseudo partial 2-trees and outerplanar graphs are $(1,3)$-pseudo partial 1-trees, our techniques yield $132\text{-}\chi_g(G) \leqslant 3$ for planar graphs G and $30\text{-}\chi_g(H) \leqslant 2$ for outerplanar graphs H. This gives a positive answer to the extension of Question 1.1 to the class of outerplanar graphs. With extra effort we can considerably improve these defects for planar and outerplanar graphs; however, these results are best possible in the sense that the right-hand side cannot be lowered, even in the non-game setting: For all d, there exist a planar graph G and an outerplanar graph H such that $d\text{-}\chi(G) > 2$ and $d\text{-}\chi(H) > 1$.

2. Warm-up

In this section we warm-up by giving a short proof that Alice can win the $(2,3)$-relaxed coloring game on a tree $T = (V,E)$. This proof was the motivation for the more complicated strategy and proofs that follow.

Assume that T has a root r and orient all edges of T toward the root. The unique outneighbor of a vertex x is called the *parent* of x and is denoted by $p(x)$. The inneighbors of x are called *sons* of x. At a given time in the game, V will be partitioned into colored vertices C and uncolored vertices U. In addition, there will be a set A of active vertices such that $C \subseteq A$. Initially $C = A = \emptyset$. Whenever a vertex is colored it is added to C; whenever it is activated or colored it is added to A.

On her first move Alice colors the root with either color. Now consider the situation later in the game after Bob has just colored a vertex x. Alice's activation strategy has two stages. In the first stage, she searches for a vertex u to color. In the second stage she colors u with a color that has not been used on its parent. If she has a choice, she chooses a color that has been used on as few sons of u as possible. It remains to describe the search stage, which has two steps, an initial step and a recursive step. Alice begins the search stage with the initial step by considering $p(x)$. If $p(x)$ is inactive, i.e. $p(x) \notin A$, then Alice sets $x := p(x)$ and moves to the recursive step. If $p(x)$ is active, but uncolored, Alice sets $u := p(x)$ and moves to the coloring stage. If $p(x)$ is colored with the same color as x and $p(p(x))$ is uncolored, then Alice sets $u := p(p(x))$ and moves to the coloring stage. (This conditional step is the extension of the usual activation strategy that leads to success in the relaxed game.) Otherwise, Alice lets u be any uncolored vertex whose parent is already colored and moves to the coloring stage. In the recursive step, Alice first activates x. If $p(x)$ is colored, then she sets $u := x$ and moves to the coloring stage. If $p(x)$ is active, but uncolored, she sets $u := p(x)$ and moves to the coloring stage. Otherwise she sets $x := p(x)$ and repeats the recursive step. We now prove:

Theorem 2.1. *If Alice follows the activation strategy, then she will win the* $(2,3)$*-relaxed coloring game.*

Proof. We begin by proving two lemmas.

Lemma 2.2. *If Alice uses the activation strategy, then at any time, any uncolored vertex u will have at most two active sons. Moreover, if u has two active sons, then Alice will immediately color u.*

Proof. Suppose that x and y are the first and second sons of u to be activated. When x was activated, Alice must have immediately activated u. After y is activated Alice will immediately color u. $\square$

Lemma 2.3. *Suppose Alice uses the activation strategy. If u is an uncolored vertex and x is a son of u that has been colored with α, then x has at most two sons that have been colored with α.*

Proof. First suppose that Bob colored x. By Lemma 2.2, at this time, x has at most one active son. After Bob colors x with α, Alice will not color any sons of x with α. If Bob colors another son of x with α, then Alice will immediately color $u = p(x)$. Now suppose that Alice colored x with α. By Lemma 2.2, x has at most two active sons. Since u is uncolored, Alice had a choice of colors when choosing α for x. Thus at most one of these sons was colored α. After x is colored α, Alice will not color any son of x with α. If Bob colors a son of x with α, Alice will immediately color u. $\square$

Suppose Alice chooses to color the uncolored vertex u with α. Then $p(u)$ has not been colored with α. Using Lemma 2.2, u has at most two sons colored with α. So coloring u will not violate Condition (1). Next consider a son x of u that has already been colored α. By Lemma 2.3, x has at most two sons colored with α. Thus coloring u with α will not violate Condition (2). Finally, we note that on any turn Bob can borrow Alice's activation strategy to find a legal move. $\square$

3. Alice's activation strategy for the relaxed coloring game

Let G be a graph with vertex set V and edge set E. Consider a linear ordering $L = v_1v_2\ldots v_n$ of V. For a vertex v_i, we denote the neighborhood of v_i by $N_G(v_i)$. We define $V^+_{G,L}(v_i) = \{v_j : j<i\}$, and $V^-_{G,L}(v_i) = \{v_j : j>i\}$. We define the set of *parents* of v by $N^+_{G,L}(v) = V^+_{G,L}(v) \cap N_G(v)$ and similarly the set of *children* of v by $N^-_{G,L}(v) = V^-_{G,L}(v) \cap N_G(v)$. We denote the *back degree* of v by $d^+_{G,L}(v) = |N^+_{G,L}(v)|$ and the *forward degree* of v by $d^-_{G,L}(v) = |N^-_{G,L}(v)|$. The *maximum back degree* of the graph G with respect to L is denoted $\Delta^+_L(G)$. Finally, let $V^+_{G,L}[v] = V^+_{G,L}(v) \cup \{v\}$, $V^-_{G,L}[v] = V^-_{G,L}(v) \cup \{v\}$, $N^+_{G,L}[v] = N^+_{G,L}(v) \cup \{v\}$ and $N^-_{G,L}[v] = N^-_{G,L}(v) \cup \{v\}$. If the graph G is clear from context, we will drop the subscript G from all notation defined above. Similarly, if the linear ordering L is clear from context, we will drop the subscript L.

Fix a linear ordering L of a graph G and suppose that Alice and Bob are playing the (r, d)-coloring game on G. During the game, vertices go from uncolored to colored. We will maintain two dynamic sets throughout. At any point in the game, let U be the set of uncolored vertices at that time and let C be the set of colored vertices. Once a vertex x is colored, we define $c(x)$ to be that color, where $c : C \to X$.

At any point in the game, we define the *defect* of a vertex $x \in C$ to be the number of vertices in $N(x)$ at that time colored with the same color as x. We denote this value by $\operatorname{def}(x)$. When the graph G is not clear from context, we will denote this value by $\operatorname{def}_G(x)$.

Suppose $x \in V$. At any point in the game, we define w to be the *mother* of x if w is the least vertex in $N^+[x]$ such that $w \in U$. We denote this vertex by $m(x)$. Note that if $x \in U$, then $m(x)$ must exist since x itself is a candidate. We also note a somewhat strange consequence of this definition: the mother of x need not be a parent of x, as x can be its own mother. We similarly define v to be the *father* of x if v is the least vertex in $N^+(x)$ such that either $v \in U$, or $c(v) = c(x)$ and $m(v)$ exists. We denote such a vertex by $f(x)$, and notice that it is possible that $f(x) = m(x)$. We also note that these definitions are dynamic. In other words, for any $x, f(x)$ and $m(x)$ may change throughout the game, and eventually, no such vertices may exist.

We shall now describe Alice's *Activation Strategy*. She first chooses a linear ordering L of V. The properties that L should satisfy will be discussed later. Alice will maintain a subset A of *active* vertices such that $C \subseteq A \subseteq V$. When a vertex x is put into A we say that x has been *activated*. Alice will only color a vertex that is already active. If Bob colors a vertex b that has not yet been activated, Alice immediately puts b into A. In this case we say that Bob, not Alice, *activated* b. Once a vertex is active, it will remain active for the remainder of the game. Initially A is empty. On her first turn, Alice activates and then colors the least vertex in L. Now suppose that Bob has just colored the vertex b. Alice must first search for the vertex she will color. Then she must decide with which color to color it.

The search stage has two parts, an initial step and a recursive step. In the initial step, Alice searches for $f(b)$. If $f(b)$ does not exist, Alice selects the least uncolored vertex u in L to be colored. If it is inactive, she activates it. If $f(b)$ exists and is uncolored, Alice sets $u := f(b)$ and moves to the recursive step. Note that in this case $f(b) = m(b)$. If $f(b)$ exists and is colored, Alice sets $u := m(f(b))$ and moves to the recursive step in her strategy.

In the recursive step, Alice considers an uncolored vertex u. If u is inactive, then Alice activates u, and finds $m(u)$. (Note that $m(u)$ might be u.) She then sets $u := m(u)$ and repeats the recursive step for this new value of u. If u is active, Alice selects u to be colored.

In the coloring stage, Alice chooses a color for u. Call a color α *eligible for* u if α has not yet been used on any parent of u. We note that as long as we choose L such that $\Delta^+(G) < |X|$, then any uncolored vertex has at least one eligible color. Alice chooses an eligible color for u that minimizes $\operatorname{def}(u)$.

When Alice moves from vertex b to consider $m(f(b))$ in the initial step of her search, we say that Alice is *skipping*, since she is skipping over the vertex $f(b)$. When she moves from a vertex u to $m(u)$, we say that she is *jumping*. If Alice is in the initial step and $f(b)$ does not exist, we say that Alice has a *free choice*. Note that in this case she selects a vertex u with no uncolored parents. When Alice activates or colors a vertex u, we say that Alice is *taking action at* u. Whenever Alice takes action at a vertex in a set $Q \subset V$, we say that Alice is *taking action in* Q. Whenever Alice encounters a vertex u in either step of the search stage (including $f(b)$ if she skips over it in the initial step) we say that she has *reached* u. Suppose that Alice reaches u at some point in her strategy. If u is either $f(v)$ or $m(v)$ for some vertex v and v has been reached on the same turn, we say that Alice reaches u *in response to* v.

4. Partial k-trees

In this section we will analyze Alice's Activation Strategy for the (r, d)-relaxed coloring game on partial k-trees. Recall that a graph H is chordal if it has no chordless cycles. A graph G is a partial k-tree if G is a subgraph of a chordal graph H, where $\omega(H) = k + 1$. It is well known that if H is chordal, then H has a *simplicial elimination ordering* (or a *simplicial ordering*). That is, there exists an ordering L of the vertices of H such that for each vertex v in the ordering, $N^+[v]$ is a clique. So with this linear ordering, $\Delta^+(H) = k$. For an arbitrary partial k-tree G, using the simplicial ordering L of a chordal supergraph H with $\omega(H) = k + 1$, we have that $\Delta^+(G) \leqslant k$.

Theorem 4.1. *If G is a partial k-tree, then $(4k-1)$-$\chi_g(G) \leqslant k+1$. Moreover, d-$\chi_g(G) \leqslant k + 1$ for all $d \geqslant 4k - 1$.*

Proof. Fix $d \geqslant 4k - 1$. First suppose that the $(k + 1, d)$-coloring game is being played on a chordal graph G. Recall that the Activation Strategy always requires Alice to color an uncolored vertex with an eligible color. We will show that if Alice follows the Activation Strategy then at any time in the game, either player can legally color any uncolored vertex u with any color that is eligible for u. Thus, eventually the entire graph will be legally colored and Alice will win. We do this with two lemmas. The first lemma states that any uncolored vertex u has at most $2k + 1$ children that are colored with colors eligible for u. Thus, it is possible to choose an eligible color for u such that $\mathrm{def}(u) \leqslant 2k + 1 \leqslant d$. The second lemma states that if a vertex x with $\mathrm{def}(x) \geqslant 4k - 1$ has been colored α, then α is not an eligible color for any uncolored parent of x. Of course, α is not an eligible color for any uncolored child of x. Thus, coloring an uncolored vertex with an eligible color cannot increase the defect of any vertex x that already has defect at least $4k - 1 \leqslant d$.

Now suppose that G is a partial k-tree. Then G has a chordal supergraph H with $\omega(H) = k + 1$. Alice will play the $(k + 1, d)$-coloring game on G as though she were playing the $(k + 1, |H|)$-coloring game on H. At any time in the game either player will be able to color any uncolored vertex u with any eligible color α so that:

1. $\mathrm{def}_G(u) \leqslant \mathrm{def}_H(u) \leqslant 4k - 1$ and

2. if v is already colored α and $\operatorname{def}_G(v) \geqslant 4k-1$ (and so $\operatorname{def}_H(v) \geqslant 4k-1$), then $\operatorname{def}_G(v)$ does not increase.

Thus eventually G will be legally colored and Alice will win. So it suffices to prove the following two lemmas.

Lemma 4.2. *Suppose that Alice follows the Activation Strategy. Then at any time, any uncolored vertex u has at most $2k+1$ children colored with colors eligible for u.*

Proof. Consider a time in the game when a vertex u is uncolored. At this time, let S be the subset of $N^-(u)$ consisting of active uncolored vertices and vertices colored with colors eligible for u. Notice that it will suffice to show that $|S| \leqslant 2k+1$. We will show that for each vertex in S there exists a unique action taken by Alice in the set $N^+[u]$. Let $x \in S$. Consider the time that x became active. Alice will next jump or skip from x. Since u is a candidate for both $f(x)$ and $m(x)$, both $f(x)$ and $m(x)$ are in $V^+[u]$. Since L is a simplicial ordering, $N^+(x)$ is a clique. Thus, since u, $f(x)$, and $m(x)$ are all in $N^+(x), f(x)$ and $m(x)$ are in $N^+[u]$. By our choice of S, x has not been colored with a color not eligible for u. Thus x has not been colored with a color used on any vertex in $N^+(u)$. So we have that $f(x) = m(x)$ and Alice will now jump to $m(x)$. Thus each time a vertex in S is activated, Alice will take action in $N^+[u]$. Since a vertex can be activated only once and colored only once, Alice can take action in $N^+[u]$ at most $2|N^+[u]|$ times. So at first we see that $|S| \leqslant 2(k+1) = 2k+2$. Moreover, consider the time that u was activated. If at this time $u = m(u)$, then Alice takes action at u by coloring u. If $u \neq m(u)$, then Alice will jump to $m(u) \in N^+(u)$. In either case, she has taken action twice in $N^+[u]$ on the same turn. Thus we have overcounted in our estimate of $|S|$ by at least one. Therefore, $|S| \leqslant 2k+1$, as desired. $\square$

Lemma 4.3. *Suppose that Alice follows the Activation Strategy. At any time, if a vertex x has been colored α and* $\operatorname{def}(x) \geqslant 4k-1$, *then α is not eligible for any uncolored parent of x.*

Proof. Fix a vertex x that has been colored with α and suppose that u is an uncolored parent of x for which α is an eligible color. It suffices to show that $\operatorname{def}(x) \leqslant 4k-2$. Let $T = N^-(u) \cap N^+(x)$, $t = |T|$, $T^- = T \cup \{x\}$, and $T^+ = T \cup \{u\}$. Observe that by the way we have chosen L, $0 \leqslant t \leqslant k-1$. Let S be the subset of $N^-(x)$, containing all vertices which are colored α at this time, or that were activated before x was colored. Let

$$S^- = \{v \in S \colon v \text{ is active before } x \text{ is colored}\}$$

and

$$S^+ = \{v \in S \colon v \text{ colored } \alpha \text{ after } x \text{ is colored}\}.$$

Clearly $S = S^- \cup S^+$. Let $s = |S|$. Since α is eligible for u, no parents of u have been colored with α. Thus, the only vertices in $N^+(x)$ which can add to the defect of x are the elements of T. So

$$\operatorname{def}(x) \leqslant |S| + |T| = s + t.$$

We will begin by overestimating this quantity and then improve the estimate with a sequence of refinements.

Let u' be the least element in T^-. Let $Q = N^+(u') \cup N^+[x]$ and $q = |Q|$. Using this Q we define the following two sets:

$$Q_A = \{v \in Q\text{: } v \text{ was activated by Alice}\}$$

and

$$Q_C = \{v \in Q\text{: } v \text{ was colored by Alice}\}.$$

Note that these two sets need not be disjoint. Let $Q^* = Q_A \cup Q_C$. We will show that for each element of S there exists a unique action taken in Q^* by Alice. Let $q^* = |Q_A| + |Q_C|$. So $q^* \leqslant 2q$.

Let $y \in S^-$. Consider the time that y became active. Alice will now skip or jump from y. First suppose that Alice skips. She will skip to $m(f(y))$ and take action at this vertex. As before, $f(y) \in N^+[x]$. By the way we have chosen S, it must be the case that $c(y) = c(f(y)) = \alpha$. Since α is not used on a parent of u, then $f(y) \in T^-$. Note that u' is a parent of each element in $T^- - \{u'\}$, implying that $m(f(y))$ is adjacent or equal to u'. This implies that $m(f(y)) \in Q^*$. On the other hand, suppose Alice jumps from y. She will jump to $m(y)$ and take action at this vertex. Since x is a candidate for $m(y)$, then as before, $m(y) \in N^+[x]$. Hence Alice will respond by taking action in Q^*. Thus, every time a vertex in S^- is activated, Alice takes action in Q^*.

Now let $z \in S^+ - S^-$. Consider the time that z was colored α. Since x is already colored α and x is a parent of z, it must be Bob who has colored z. So in the initial step in her strategy, Alice searches for $f(z)$. Since x itself is a candidate, $f(z) \in N^+[x]$. If $f(z)$ is colored, it is colored with α. Thus $f(z) \in T^-$. So u is a candidate for $m(f(z))$. Thus $m(f(z)) \in V^+[u]$. Also $m(f(z))$ is adjacent to u', so $m(f(z)) \in Q^*$. If $f(z)$ is uncolored, then Alice will take action at $f(z)$. Thus, in either case, Alice takes action in Q^*. We have now shown that each element in $S^+ - S^-$ corresponds to a unique action by Alice in Q^*.

As before, since Alice can activate a vertex at most once and color a vertex at most once, we have that $s \leqslant q^*$. But observe that $u \in N^+(u') \cap N^+[x]$, implying that

$$q \leqslant |N^+(u')| + |N^+[x]| - |N^+(u') \cap N^+[x]| \leqslant k + (k+1) - 1 = 2k.$$

Thus, we have that

$$\operatorname{def}(x) \leqslant s + t \leqslant 2(2k) + (k-1) = 5k - 1.$$

We can improve this. Consider a vertex $z \in T^-$ that Alice activates in response to a vertex in S. She will now jump to $m(z)$. Observe that u is a candidate for $m(z)$ and $u' \in N^+[z]$. Thus $m(z) \in N^+(u')$. Thus Alice has taken action twice in Q^* on the same turn. This gives the refined estimate

$$s \leqslant q^* - |T^-| \leqslant 2q - |T^-|.$$

Thus,

$$\mathrm{def}(x) \leqslant s + t \leqslant (2q - |T^-|) + |T| \leqslant 4k - 1.$$

We can now make one additional refinement by considering who colored the vertex x. First suppose that Bob colored x. Alice would respond by taking action at a vertex in $Q^* - \{x\}$. But this was already counted in our estimate of s. Thus, we have overcounted by at least one.

Now suppose that Alice colored x. Note that at the time that she colored x, u was uncolored. Therefore, she has a choice of at least two eligible colors with which to color x, since x could have at most $k-1$ colored parents. Since she chose to color x with α, it was either because x had no colored children and the choice was arbitrary, or because x had a child colored β for some eligible color β with $\alpha \neq \beta$.

First suppose that x has no colored children at this time. Consider the turn on which Alice activated x. She must have jumped to another vertex after activating x. Then $x = m(v)$ for some v on this turn. Since x has no colored children at the time that it was colored, v must have been uncolored at the time that x was activated. Now, if v eventually gets colored α, then it will be after x is colored implying that Alice did not color v. Then when Bob colors v, Alice will take action in $Q^* - \{x\}$, since x is a candidate for $f(v)$. Otherwise v does not get colored α and so it will not be counted in $\mathrm{def}(x)$, although we have counted the action in Q^* taken by Alice when v was activated in our estimate of s. In either case we have overestimated s by at least one.

On the other hand, suppose that x had a child w colored β at the time that Alice colors x. Consider the time when w became active. Either Bob has just colored w, or Alice reaches w by skipping or jumping. If Bob has just colored w, Alice will find $f(w)$. Observe that $f(w)$ exists, as x itself is a candidate. Thus $f(w) \in N^+[x]$. Recall that since β is a candidate for the color that Alice is choosing for x, β is not used on any vertex in $N^+(x)$. This implies that Alice will not skip on this turn. Hence, Alice takes action in Q^*. If Alice reaches w by skipping or jumping, she will seek $m(w)$. But again, x is a candidate for $m(w)$ implying that Alice will take action in Q^*. So in either case, w is not counted in $\mathrm{def}(x)$, although we have counted it in our estimate of s.

Therefore, no matter who colored x, we have overestimated $\mathrm{def}(x)$ by at least one. Thus we have $\mathrm{def}(x) \leqslant 4k - 2$, as desired. □

The combination of these two lemmas then completes the proof of the theorem. □

We can now derive corollaries from Theorem 4.1 for specific classes of partial k-trees.

Corollary 4.4. *If G is outerplanar, then* $7\text{-}\chi_g(G) \leqslant 3$.

Corollary 4.5. *If G is a tree, then* $3\text{-}\chi_g(G) = 2$.

5. Graphs with bounded k-admissability

Let $G=(V,E)$ be a graph and suppose that there exists a linear ordering L of V such that with this L, $\Delta^{+}(G)\leqslant k$. For each $x\in L$, we say that a subset $T\subset V$ is *k-admissable for x relative to L* if the following two conditions hold:

1. $y\leqslant x$ in L for all $y\in T$, and
2. there exists an injection $\theta: T\to V$, such that $y\theta(y)\in E$, $x\theta(y)\in E$, and $x<\theta(y)$ in L for all $y\in T$.

The *k-admissability of x relative to L* is the maximum $|T|$ where T is k-admissable for x relative to L. The *k-admissability of L* is the maximum value of the k-admissability of all of the vertices relative to L. We say that a graph has *k-admissability a* if there exists a linear ordering L with $\Delta^{+}(G)\leqslant k$, where L has k-admissability at most a. Note that k-admissability is almost the same as *admissability*, as defined by Kierstead and Trotter in [14]. The only differences are that we are restricting ourselves to linear orderings with back degree at most k, and we do not allow vertices in T solely because they are in $N^{+}(x)$. The notion of admissibility in turn was heavily influenced by the definition of *arrangability* introduced by Chen and Schelp [3] to attack a Ramsey theoretic problem.

We are now ready to show that there exists a function $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that if G has k-admissability at most a, then $f(a,k)$-$\chi_g(G)\leqslant k+1$.

Theorem 5.1. *Let G be a graph with k-admissability a. If $d\geqslant(2k+1)a+k$, then d-$\chi_g(G)\leqslant k+1$.*

Proof. Suppose that G has k-admissability a, and that the linear ordering L is chosen to witness this. Alice will use the Activation Strategy with respect to L. We will show that at any time any uncolored vertex u has at most $(2k+1)a$ children colored and that if x is a colored vertex with an uncolored parent, then $\mathrm{def}(x)\leqslant(2k+1)a+k-1$. First we will need two lemmas.

Lemma 5.2. *Suppose that Alice follows the Activation Strategy. Then she will never reach the same vertex more than $2k+1$ times in response to other vertices.*

Proof. Consider a vertex v. Each time Alice reaches v she will either take action at v or skip over v to $m(v)\in N^{+}(v)$, and then take action at $m(v)$. Since she can take action at a vertex at most twice, she can reach v at most $2(k+1)$ times. Moreover, the first time she takes action at v, if ever, she will activate v and then take another action at $m(v)\in N^{+}[v]$. Thus, we have overestimated by at least one. (Note that if $m(v)=v$ when v is activated, then v may be reached by Alice $2k+2$ times, but one of these times will be from v.) $\square$

Lemma 5.3. *Suppose that Alice follows the Activation Strategy and that at some time x is a vertex whose closed neighborhood $N^+[x]$ contains an uncolored vertex u. If x is colored, let $P = \{c(x)\}$; otherwise let $P = X$, the set of all colors. Then x has at most $(2k+1)a$ children colored with colors from P.*

Proof. Let $S \subset N^-(x)$ be the subset consisting of all children of x which are either colored with a color in P or which were activated before x was colored. It suffices to show that $|S| \leqslant (2k+1)a$. For each $y \in S$, let $p(y)$ be the next vertex that Alice reaches in response to y being activated. Then $p(y) \in V^+[x]$: If we are at a point in the game when x is uncolored, then x is a candidate for $m(y)$ and $f(y)$. If x is colored, then y must have been colored $c(x)$ by Bob. In this case x is a candidate for $f(y)$ since u is uncolored, and therefore a candidate for $m(x)$. Let $T = \{p(y) : y \in S\}$. For each vertex $v \in T$, let $\theta(v)$ be the first preimage of v under p to be activated. Clearly θ witnesses that T is a k-admissible set for x relative to L. Thus $|T| \leqslant a$.

Now, for each $v \in T$, we define

$$S_v = \{w \in S \colon p(w) = v\}.$$

So by definition, the collection $\{S_v\}_{v \in T}$ partitions S. Thus, we have that

$$|S| = \sum_{v \in T} |S_v|$$

By Lemma 5.2, Alice can reach v at most $2k+1$ times in response to other vertices. Thus $|S_v| \leqslant 2k+1$. Hence,

$$|S| = \sum_{v \in T} |S_v| \leqslant (2k+1)a. \qquad \Box$$

Now suppose Alice chooses to color the uncolored vertex u. Since $u \in N^+[u]$, by Lemma 5.3, u has at most $(2k+1)a$ colored children. Alice will color u with a color not used on any of the parents of u, so $\mathrm{def}(u) \leqslant (2k+1)a$. This will not increase the defect of any parent of u. If x is a child of u, then $u \in N^+[x]$, so by Lemma 5.3, $\mathrm{def}(x) \leqslant (2k+1)a + k$. $\Box$

In [14] it was shown that all planar graphs have admissability at most 8. With a little work reconciling the definitions of admissability and k-admissability, it is not difficult to see that this implies that planar graphs have 5-admissability at most 8. We then have the following corollary to Theorem 5.1.

Corollary 5.4. *If G is planar then d-$\chi_g(G) \leqslant 6$ for all $d \geqslant 93$.*

6. Open questions

In this article and its sequel [9] primarily our motivation has been the Question 1.1 raised in [9]: Does there exist d such that d-$\chi_g(T) \leqslant 2$ for all trees T? In [9] it is shown that $d = 1$ is too small and we have shown that any $d \geqslant 3$ works. Since trees have been a useful playground for developing general strategies for game coloring, we believe the following obvious question should be answered.

Question 6.1 (Chou et al. [4]). Does Alice have a winning strategy for the $(2,2)$-relaxed coloring game on trees?

The solution to Question 1.1 led to several extensions of the Activation Strategy for answering questions of the following form: For a fixed class of graphs $\mathscr{G}$ and integer r, does there exist a defect d such that $d\text{-}\chi_g(G) \leqslant r$ for all graphs $G \in \mathscr{G}$? Chou et al. [4] attacked a different question. They investigated upper bounds on $1\text{-}\chi_g(G)$ and showed that $1\text{-}\chi_g(T) \leqslant 3$ and $1\text{-}\chi_g(H) \leqslant 6$, for trees T and outerplanar graphs H. As mentioned above, they also showed that the bound for trees is best possible. Thus they fixed a defect d and studied bounds on $d\text{-}\chi_g(G)$.

Question 6.2. What are the best upper bounds for $1\text{-}\chi_g(G)$ for chordal, outerplanar, and planar graphs G?

In [4] the upperbounds for trees are obtained with a separator strategy, while the upper bounds for outerplanar graphs are obtained with an Activation Strategy. We can duplicate the result for trees using an Activation Strategy. This gives hope for attacking the problem on planar graphs. On the other hand, a separator strategy, as used in [16], might yield results for chordal graphs.

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